Example 1

Solve a Quadratic Inequality of the Form $ax^2 + bx + c \leq 0$, $a > 0$

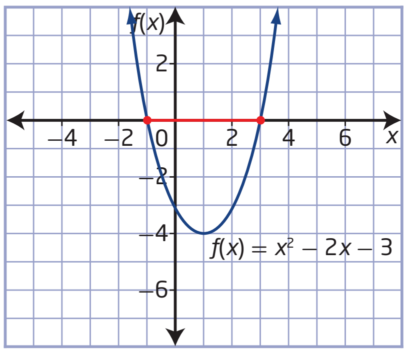
Solve $x^2 - 2x - 3 \leq 0$.

Solution

Method 1: Graph the Corresponding Function

Graph the corresponding function $f(x) = x^2 - 2x - 3$.

To determine the solution to $x^2 - 2x - 3 \leq 0$, look for the values of x for which the graph of $f(x)$ lies on or below the x-axis.



What strategies can you use to sketch the graph of a quadratic function in standard form?

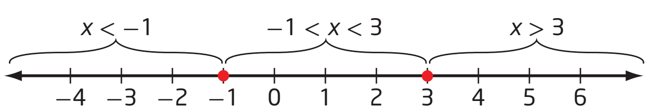
The parabola lies on the x-axis at $x = -1$ and $x = 3$. The graph lies below the x-axis between these values of x. Therefore, the solution set is all real values of x between -1 and 3, inclusive, or $\{x \mid -1 \leq x \leq 3, x \in \mathrm{R}\}$.

Method 2: Roots and Test Points

Solve the related equation $x^2 - 2x - 3 = 0$ to find the roots. Then, use a number line and test points to determine the intervals that satisfy the inequality.

$$x^2 - 2x - 3 = 0$$
$$(x + 1)(x - 3) = 0$$
$$x + 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -1 \qquad\qquad x = 3$$

Plot -1 and 3 on a number line. Use closed circles since these values are solutions to the inequality.



Does it matter which values you choose as test points?

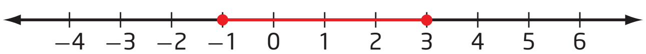
Are there any values that you should not choose?

The x-axis is divided into three intervals by the roots of the equation. Choose one test point from each interval, say -2, 0, and 5. Then, substitute each value into the quadratic inequality to determine whether the result satisfies the inequality.

Use a table to organize the results.

Interval	$x < -1$	$-1 < x < 3$	$x > 3$
Test Point	-2	0	5
Substitution	$(-2)^2 - 2(-2) - 3$ $= 4 + 4 - 3$ $= 5$	$0^2 - 2(0) - 3$ $= 0 + 0 - 3$ $= -3$	$5^2 - 2(5) - 3$ $= 25 - 10 - 3$ $= 12$
Is $x^2 - 2x - 3 \leq 0$?	no	yes	no

The values of x between -1 and 3 also satisfy the inequality. The value of $x^2 - 2x - 3$ is negative in the interval $-1 < x < 3$. The solution set is $\{x \mid -1 \leq x \leq 3, x \in \mathrm{R}\}$.



Method 3: Case Analysis

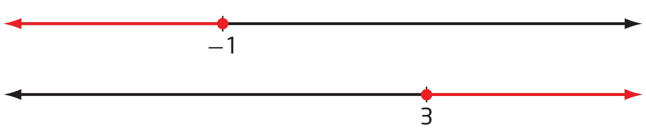
Factor the quadratic expression to rewrite the inequality as $(x + 1)(x - 3) \leq 0$.

The product of two factors is negative when the factors have different signs. There are two ways for this to happen.

Case 1: The first factor is negative and the second factor is positive.

$x + 1 \leq 0$ and $x - 3 \geq 0$

Solve these inequalities to obtain $x \leq -1$ and $x \geq 3$.



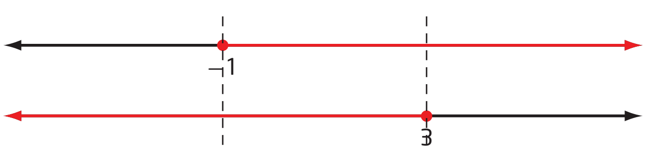
Any x-values that satisfy both conditions are part of the solution set. There are no values that make both of these inequalities true.

Why are there no values that make both inequalities true?

Case 2: The first factor is positive and the second factor is negative.

$x + 1 \geq 0$ and $x - 3 \leq 0$

Solve these inequalities to obtain $x \geq -1$ and $x \leq 3$.



The dashed lines indicate that $-1 \leq x \leq 3$ is common to both.

These inequalities are both true for all values between -1 and 3, inclusive.

The solution set is $\{x \mid -1 \leq x \leq 3, x \in \mathrm{R}\}$.

How would the steps in this method change if the original inequality were $x^2 - 2x - 3 \geq 0$?

Your Turn

Solve $x^2 - 10x + 16 \leq 0$ using two different methods.

Example 2

Solve a Quadratic Inequality of the Form $ax^2 + bx + c < 0$, $a < 0$

Solve $-x^2 + x + 12 < 0$.

Solution

Method 1: Roots and Test Points

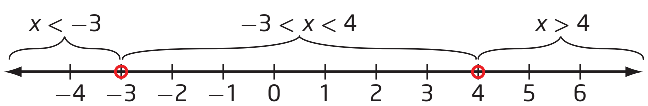
Solve the related equation $-x^2 + x + 12 = 0$ to find the roots.

$$\begin{aligned}
-x^2 + x + 12 &= 0 \\
-1(x^2 - x - 12) &= 0 \\
-1(x + 3)(x - 4) &= 0
\end{aligned}$$

$$x + 3 = 0 \quad \text{or} \quad x - 4 = 0$$
$$x = -3 \qquad\qquad x = 4$$

Plot -3 and 4 on a number line.

Use open circles, since these values are not solutions to the inequality.



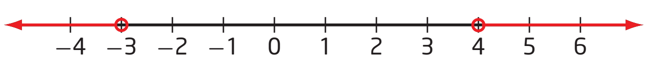
Choose a test point from each of the three intervals, say -5, 0, and 5, to determine whether the result satisfies the quadratic inequality.

Use a table to organize the results.

Interval	$x < -3$	$-3 < x < 4$	$x > 4$
Test Point	-5	0	5
Substitution	$-(-5)^2 + (-5) + 12$ $= -25 - 5 + 12$ $= -18$	$-0^2 + 0 + 12$ $= 0 + 0 + 12$ $= 12$	$-5^2 + 5 + 12$ $= -25 + 5 + 12$ $= -8$
Is $-x^2 + x + 12 < 0$?	yes	no	yes

The values of x less than -3 or greater than 4 satisfy the inequality.

The solution set is $\{x \mid x < -3 \text{ or } x > 4, x \in \mathrm{R}\}$.



Method 2: Sign Analysis

Factor the quadratic expression to rewrite the inequality as $-1(x + 3)(x - 4) < 0$.

Determine when each of the factors, $-1(x + 3)$ and $x + 4$, is positive, zero, or negative.

Since -1 is a constant factor, combine it with $(x + 3)$ to form one factor.

Substituting -4 in $-1(x + 3)$ results in a positive value (+).

$$\begin{aligned} -1(-4 + 3) &= -1(-1) \\ &= 1 \end{aligned}$$

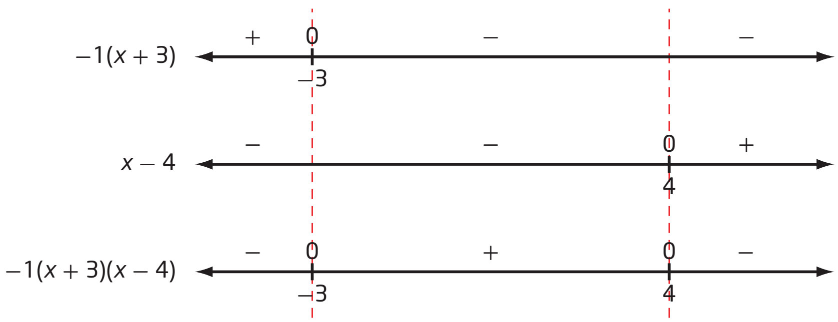
Substituting -3 in $-1(x + 3)$ results in a value of zero (0).

$$\begin{aligned} -1(-3 + 3) &= -1(0) \\ &= 0 \end{aligned}$$

Substituting 1 in $-1(x + 3)$ results in a negative value (−).

$$\begin{aligned} -1(1 + 3) &= -1(4) \\ &= 1 \end{aligned}$$

Sketch number lines to show the results.



From the number line representing the product, the values of x less than -3 or greater than 4 satisfy the inequality $-1(x + 3)(x - 4) < 0$.

The solution set is $\{x \mid x < -3 \text{ or } x > 4, x \in \mathrm{R}\}$.

Your Turn

Solve $-x^2 + 3x + 10 < 0$ using two different methods.