Completing the Square

Working Example 1: Convert from Standard Form to Vertex Form

Rewrite each function in vertex form by completing the square. State the vertex for each.

a) $y = x^2 - 8x + 13$

b) $y = -2x^2 + 12x + 2$

c) $y = 3x^2 + 2x - 1$

Solution

a) Group the first two terms.

$y = (x^2 - 8x) + 13$

Then, add and subtract the square of half the coefficient of the x-term to create a perfect square trinomial.

$y = (x^2 - 8x + 16 - 16) + 13$

$\left(\frac{-8}{2}\right)^2 = (-4)^2$
$= 16$

Factor the first three terms, which will always be a perfect square trinomial.

$y = (x - 4)^2 - 16 + 13$

Simplify.

$y = (x - 4)^2 - 3$

The vertex is at (4, −3).

b) After grouping the first two terms, factor out the coefficient −2 from the group.

$y = -2(x^2 - 6x) + 2$

Complete the square as in part a).

$y = -2(x^2 - 6x +$ ______ $-$ ______$) + 2$

$y = -2[(x -$ ______$)^2 -$ ______$] + 2$

$y = -2(x -$ ______$)^2 + 18 + 2$

$y = -2(x -$ ______$)^2 +$ ______

Determine the quantity to be added and subtracted by calculating the square of half the coefficient of the x-term.

Remember that the distributive property applies to the fourth term in the parentheses.

The vertex is at (3, 20).

c) Though 2 is not a factor of 3, you can still begin by grouping and factoring.

$y = 3\left(x^2 + \frac{2}{3}x\right) - 1$

$y = 3\left(x^2 + \frac{2}{3}x +\right.$ ______ $-$ ______$\left.\right) - 1$

$y = 3[(x +$ ______$)^2 -$ ______$] - 1$

$y = 3(x +$ ______$)^2 -$ ______ $- 1$

$y = 3(x +$ ______$)^2 -$ ______

The vertex is at ______________.

Working Example 2: Convert to Vertex Form and Verify

a) Convert the function $y = -3x^2 - 24x - 19$ to vertex form.

b) Verify that the two forms are equivalent.

Solution

a) Complete the square for the function $y = -3x^2 - 24x - 19$.

$y = (\underline{\qquad} - \underline{\qquad}) - \underline{\qquad}$

$y = -3(\underline{\qquad} + \underline{\qquad}) - \underline{\qquad}$

$y = -3[(x + \underline{\qquad})^2 - \underline{\qquad}] - \underline{\qquad}$

$y = -3(x + \underline{\qquad})^2 + \underline{\qquad} - \underline{\qquad}$

$y = \underline{\qquad\qquad}$

b) **Method 1: Use Algebra**

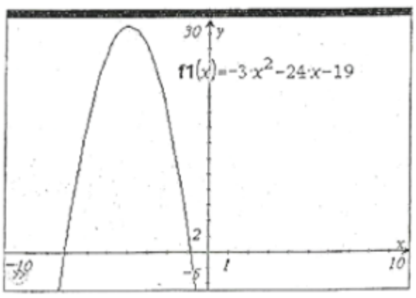
To verify that the two forms are equivalent, expand and simplify the vertex form of the function.

$y = -3(x + 4)^2 + 29$
$y = -3(x + 4)(x + 4) + 29$
$y = -3(x^2 + 4x + 4x + 16) + 29$
$y = -3(x^2 + 8x + 16) + 29$
$y = -3x^2 - 24x - 48 + 29$
$y = -3x^2 - 24x - 19$

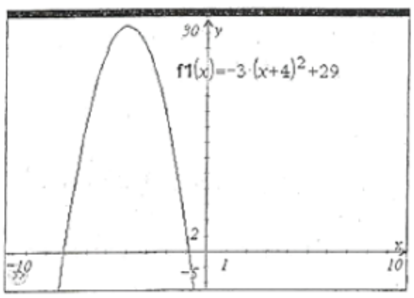
Method 2: Use Technology

Use graphing technology to graph each function.

$y = -3x^2 - 24x - 19$



$y = -3(x + 4)^2 + 29$



Since the graphs appear identical, the two forms are equivalent.

Example 2

Convert to Vertex Form and Verify

a) Convert the function $y = 4x^2 - 28x - 23$ to vertex form.

b) Verify that the two forms are equivalent.

Solution

a) Complete the square to convert to vertex form.

Method 1: Use Fractions

$$y = 4x^2 - 28x - 23$$
$$y = 4(x^2 - 7x) - 23$$
$$y = 4\left[x^2 - 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2\right] - 23$$
$$y = 4\left(x^2 - 7x + \frac{49}{4} - \frac{49}{4}\right) - 23$$
$$y = 4\left[\left(x^2 - 7x + \frac{49}{4}\right) - \frac{49}{4}\right] - 23$$
$$y = 4\left[\left(x - \frac{7}{2}\right)^2 - \frac{49}{4}\right] - 23$$
$$y = 4\left(x - \frac{7}{2}\right)^2 - 4\left(\frac{49}{4}\right) - 23$$
$$y = 4\left(x - \frac{7}{2}\right)^2 - 49 - 23$$
$$y = 4\left(x - \frac{7}{2}\right)^2 - 72$$

Why is the number being added and subtracted inside the brackets not a whole number in this case?

Method 2: Use Decimals

$y = 4x^2 - 28x - 23$
$y = 4(x^2 - 7x) - 23$
$y = 4[x^2 - 7x + (3.5)^2 - (3.5)^2] - 23$
$y = 4(x^2 - 7x + 12.25 - 12.25) - 23$
$y = 4[(x^2 - 7x + 12.25) - 12.25] - 23$
$y = 4[(x - 3.5)^2 - 12.25] - 23$
$y = 4(x - 3.5)^2 - 4(12.25) - 23$
$y = 4(x - 3.5)^2 - 49 - 23$
$y = 4(x - 3.5)^2 - 72$

Do you find it easier to complete the square using fractions or decimals? Why?

Example 3

Determine the Vertex of a Quadratic Function by Completing the Square

Consider the function $y = 5x^2 + 30x + 41$.

a) Complete the square to determine the vertex and the maximum or minimum value of the function.

b) Use the process of completing the square to verify the relationship between the value of p in vertex form and the values of a and b in standard form.

c) Use the relationship from part b) to determine the vertex of the function. Compare with your answer from part a).

Solution

a) $y = 5x^2 + 30x + 41$

$y = 5(x^2 + 6x) + 41$

$y = 5(x^2 + 6x + 9 - 9) + 41$

$y = 5[(x^2 + 6x + 9) - 9] + 41$

$y = 5[(x + 3)^2 - 9] + 41$

$y = 5(x + 3)^2 - 45 + 41$

$y = 5(x + 3)^2 - 4$

The vertex form of the function, $y = a(x - p)^2 + q$, reveals characteristics of the graph.

The vertex is located at the point (p, q). For the function $y = 5(x + 3)^2 - 4$, $p = -3$ and $q = -4$. So, the vertex is located at $(-3, -4)$. The graph opens upward since a is positive. Since the graph opens upward from the vertex, the function has a minimum value of -4 when $x = -3$.