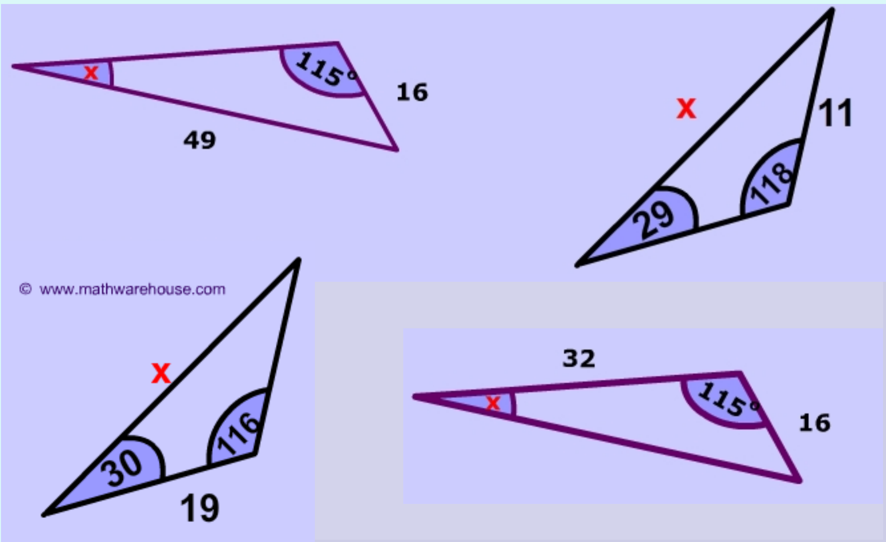
SUMMARY

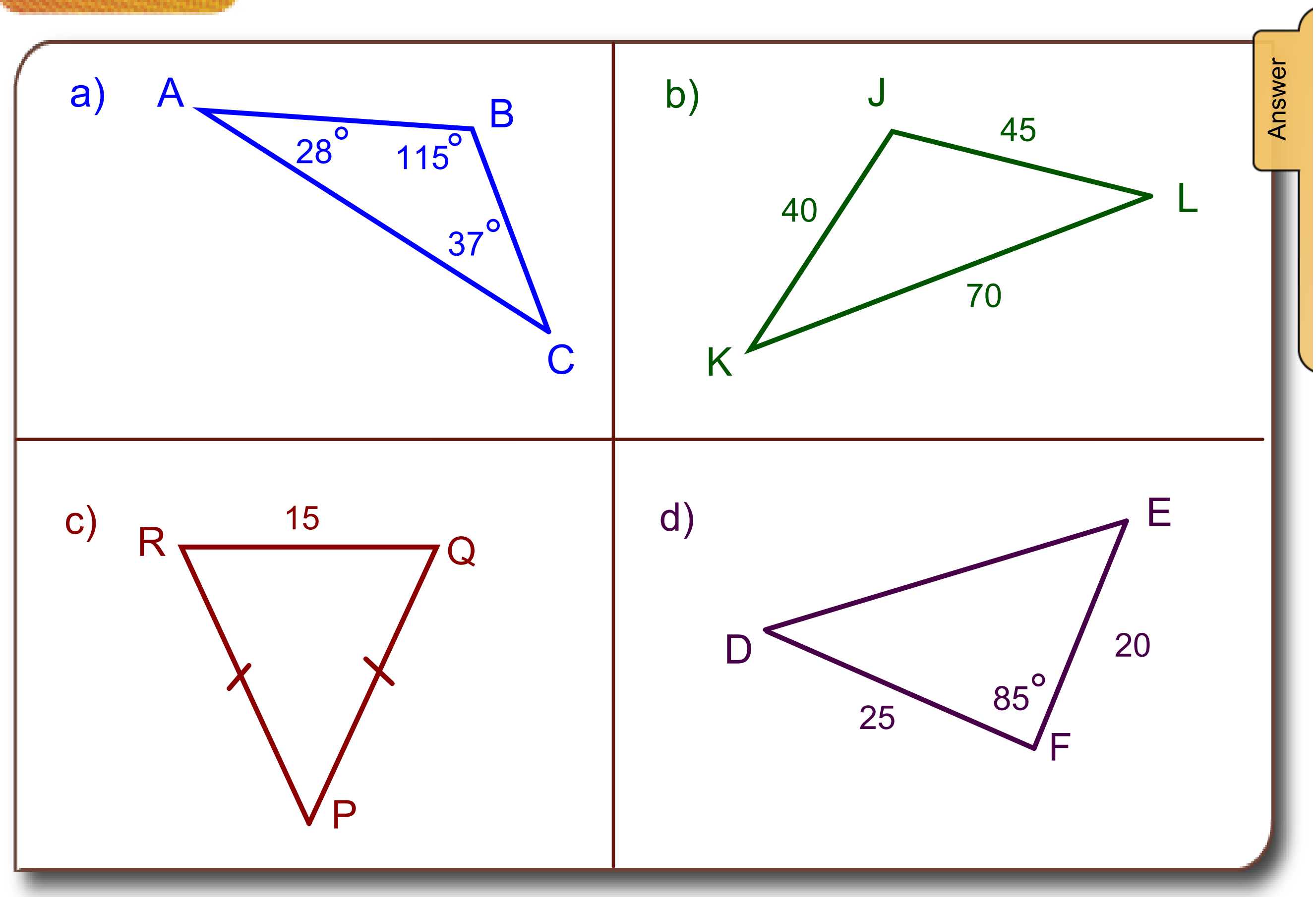
You cannot use the Law of Sines to solve all problems involving oblique triangles.



Chapter 2

The Sine Law

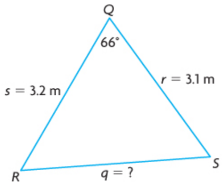
Explain why the sine law cannot be used to solve each triangle.



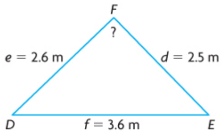
INVESTIGATE the Math

Pull for Lesson Notes

The sine law cannot always help you determine unknown angle measures or side lengths. Consider these triangles:



where $\frac{3.1}{\sin R} = \frac{3.2}{\sin S} = \frac{q}{\sin 66°}$



where $\frac{\sin E}{2.6} = \frac{\sin D}{2.5} = \frac{\sin F}{3.6}$

There are two unknowns in each pair of equivalent ratios, so the pairs cannot be used to solve for the unknowns. Another relationship is needed. This relationship is called the **cosine law**, and it is derived from the Pythagorean theorem.

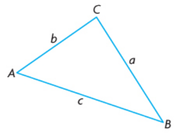
cosine law

In any acute triangle,

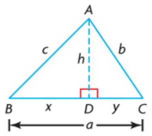
$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2ac \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$



Before this relationship can be used to solve problems, it must be proven to work in all acute triangles. Consider Heather's proof of the cosine law:



Step 1
I drew an acute triangle *ABC*. Then I drew the height from *A* to *BC* and labelled the intersection point as point *D*. I labelled this line segment *h*. I labelled *BD* as *x* and *DC* as *y*.

$h^2 = c^2 - x^2$
$h^2 = b^2 - y^2$

Step 2
I wrote two different expressions for h^2.

$c^2 - x^2 = b^2 - y^2$
$c^2 = x^2 + b^2 - y^2$

Step 3
I set the two expressions equal to each other and solved for c^2.

$x = a - y$, so
$c^2 = (a - y)^2 + b^2 - y^2$
$c^2 = a^2 - 2ay + y^2 + b^2 - y^2$
$c^2 = a^2 + b^2 - 2ay$

Step 4
I wrote an equivalent equation that only used the variable *y* and simplified.

$\cos C = \frac{y}{b}$, so
$b \cos C = y$

Step 5
I determined an equivalent expression for *y*.

$c^2 = a^2 + b^2 - 2ay$
$c^2 = a^2 + b^2 - 2ab \cos C$

Step 6
I substituted the expression $b \cos C$ for *y* in my equation.

? How can you improve Heather's explanations in her proof of the cosine law?

A. Work with a partner to explain why she drew height *AD* in step 1.

B. In step 2, Heather created two different expressions that involved h^2. Explain how she did this.

C. Explain why she was able to set the expressions for h^2 equal in step 3.

D. In step 4, Heather eliminated the variable *x*. Explain how and why.

E. Explain how she determined an equivalent expression for *y* in step 5.

F. Explain why the final equation in step 6 is the most useful form of the cosine law.

Answers

A.

B.

C.

D.

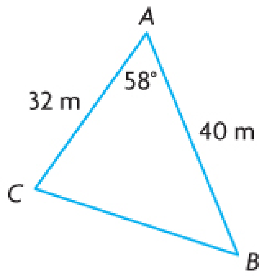
E.

F.

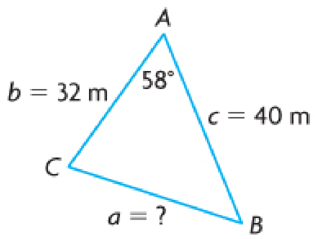
APPLY the Math

EXAMPLE 1 Using reasoning to determine the length of a side

Determine the length of CB to the nearest metre.



Justin's Solution



I labelled the sides with letters.
I couldn't use the sine law, because I didn't know a side length and the measure of its opposite angle.

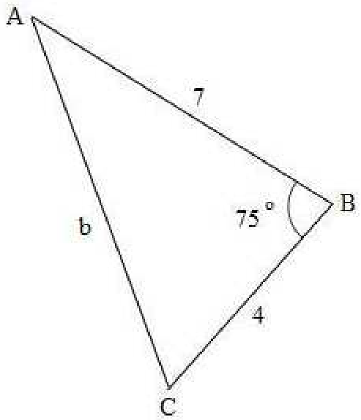
$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$a^2 = 32^2 + 40^2 - 2(32)(40) \cos 58°$$

I knew the lengths of two sides (b and c) and the measure of the contained angle between these sides ($\angle A$). I had to determine side a, which is opposite $\angle A$. I chose the form of the cosine law that includes these four values. Then I substituted the values I knew into the cosine law.

$$a^2 = 1024 + 1600 - 2560 \cos 58°$$
$$a^2 = 2624 - 2560 \cos 58°$$
$$a^2 = 1267.406...$$
$$a = \sqrt{1267.406...}$$
$$a = 35.600...$$

CB is 36 m.

What is the length of side b?



$b^2 = 4^2 + 7^2 - 2(4)(7)\cos 75^\circ$

$b^2 = 16 + 49 - 14.49...$

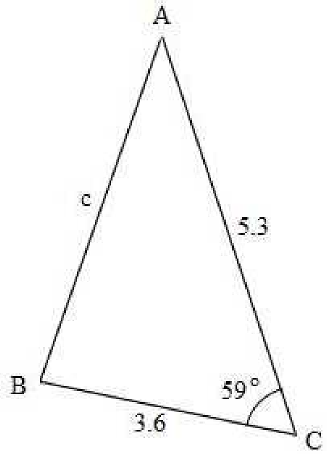
$b^2 = 65 - 14.49....$

$b^2 = 50.506....$

$b = 7.11$

A 6.36	B 7.11
C 7.78	D 8.06

What is the length of side c?



$c^2 = 3.6^2 + 5.3^2 - 2(3.6)(5.3)\cos 59^\circ$

$b^2 = 12.96 + 28.09 - 19.65...$

$b^2 = 41.05 - 19.65...$

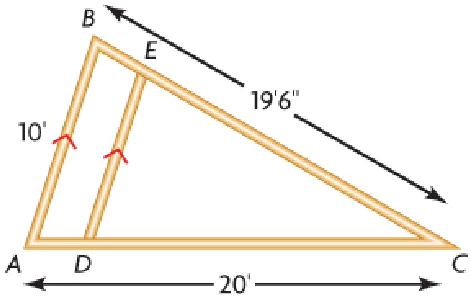
$b^2 = 21.396....$

$b = 4.63$

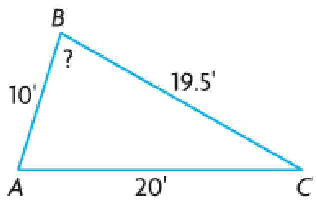
A 4.63	B 4.73
C 5.60	D 6.41

EXAMPLE 2 Using reasoning to determine the measure of an angle

The diagram at the right shows the plan for a roof, with support beam *DE* parallel to *AB*. The local building code requires the angle formed at the peak of a roof to fall within a range of 70° to 80° so that snow and ice will not build up. Will this plan pass the local building code?



Luanne's Solution: Substituting into the cosine law, then rearranging



I drew a sketch, removing the support beam since it isn't needed to solve this problem.

The peak of the roof is represented by $\angle B$.

I labelled the sides I knew in the triangle.

I wrote all the lengths using the same unit, feet.

$a = 19.5,\ b = 20$, and $c = 10$

Since I only knew the lengths of the sides in the triangle, I couldn't use the sine law.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

I had to determine $\angle B$, so I decided to use the form of the cosine law that contained $\angle B$.

$$20^2 = 19.5^2 + 10^2 - 2(19.5)(10)\cos B$$
$$400 - 380.25 - 100 = -390(\cos B)$$
$$-80.25 = -390(\cos B)$$
$$\frac{-80.25}{-390} = \cos B$$
$$\cos^{-1}\left(\frac{80.25}{390}\right) = \angle B$$
$$78.125...^\circ = \angle B$$

I substituted the side lengths into the formula and simplified.

I had to isolate $\cos B$ before I could determine $\angle B$.

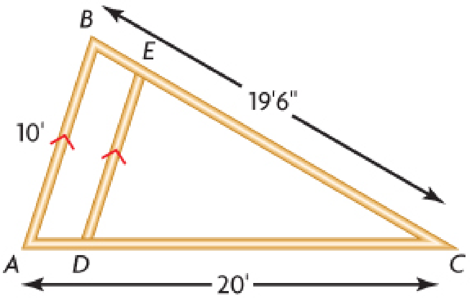
The angle formed at the peak of the roof is 78°. This plan will pass the local building code.

My answer is reasonable because $\angle B$ should be the angle with the largest measure in the triangle.

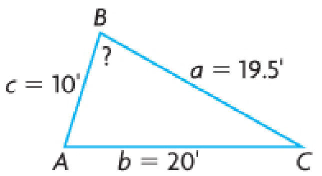
78° lies within the acceptable range of 70° to 80°.

EXAMPLE 2 Using reasoning to determine the measure of an angle

The diagram at the right shows the plan for a roof, with support beam DE parallel to AB. The local building code requires the angle formed at the peak of a roof to fall within a range of 70° to 80° so that snow and ice will not build up. Will this plan pass the local building code?



Emilie's Solution: Rearranging the cosine law before substituting



I drew a diagram, labelling the sides and angles. I wrote all the side lengths in terms of feet.

$$b^2 = a^2 + c^2 - 2ac \cos B$$

Since I wanted to determine $\angle B$ and I knew the length of all three sides, I wrote the form of the cosine law that contains $\angle B$.

$$b^2 + 2ac \cos B = a^2 + c^2 - 2ac \cos B + 2ac \cos B$$
$$b^2 + 2ac \cos B = a^2 + c^2$$
$$b^2 + 2ac \cos B - b^2 = a^2 + c^2 - b^2$$
$$2ac \cos B = a^2 + c^2 - b^2$$
$$\frac{2ac \cos B}{2ac} = \frac{a^2 + c^2 - b^2}{2ac}$$
$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

I decided to rearrange the formula to solve for $\cos B$ by adding $2ac \cos B$ to both sides of the equation. Then I subtracted b^2 from both sides. Finally I divided both sides by $2ac$.

$$\cos B = \frac{19.5^2 + 10^2 - 20^2}{2(19.5)(10)}$$

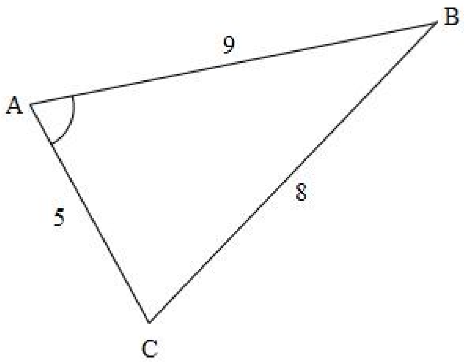
I substituted the information that I knew into the rearranged formula and evaluated the right side.

$$\cos B = \frac{80.25}{390}$$
$$\cos B = 0.2057...$$
$$\angle B = \cos^{-1}(0.2057...)$$
$$\angle B = 78.125...°$$

The angle formed at the peak of the roof is 78°. This plan passes the local building code.

I rounded to the nearest degree. The value of this angle is within the acceptable range.

Find angle A



$8^2 = 5^2 + 9^2 - 2(5)(9)\cos A$

$64 = 25 + 81 - 90\cos A$

$-42 = -90\cos A$

$\frac{-42}{-90} = \cos A$

$\cos A = 0.466....$

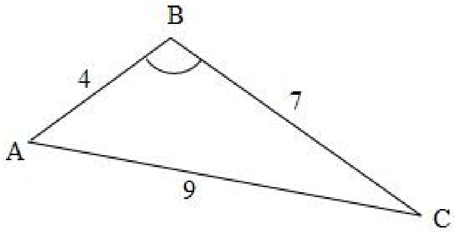
$A = 62.2°$

-90

A 33.6°	B 54.7°
C 62.2°	D 84.3°

$9^2 = 7^2 + 4^2 - 2(7)(4)\cos B$

Find angle B



$81 = 49 + 16 - 56\cos A$

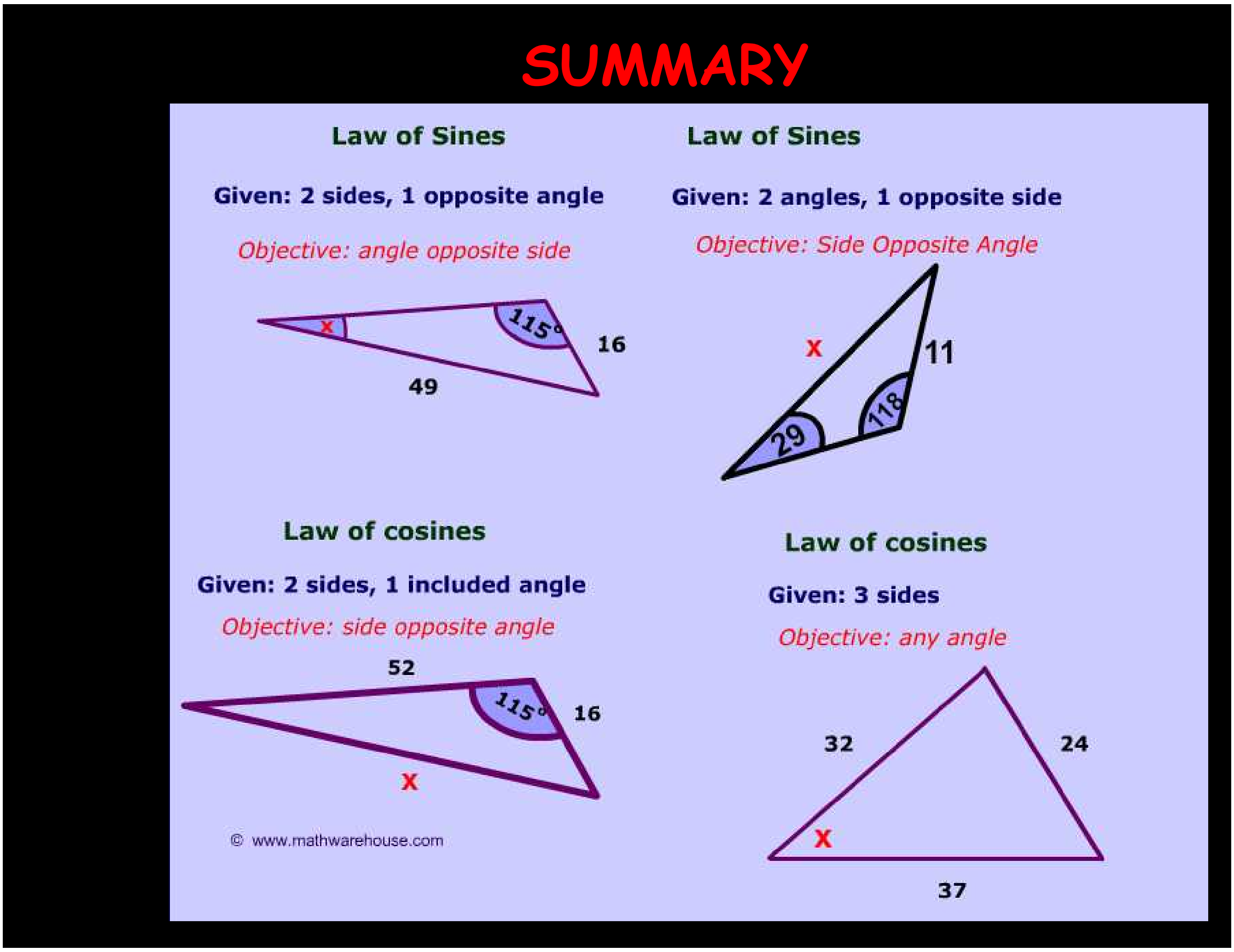
$16 = -56\cos A$

$\frac{16}{-56} = \cos A$

$\cos A = -0.285....$

$A = 106.6°$

A 73.4°	B 106.6°
C 131.8°	D 154.8°


SUMMARY
Law of Sines
Given: 2 sides, 1 opposite angle
Objective: angle opposite side
x
115°
16
49
Law of Sines
Given: 2 angles, 1 opposite side
Objective: Side Opposite Angle
x
11
29
118
Law of cosines
Given: 2 sides, 1 included angle
Objective: side opposite angle
52
115°
16
x
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Law of cosines
Given: 3 sides
Objective: any angle
32
24
x
37

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